Random Graphs and Limits | Semester I 2020/21

Back-paper Exam **Date : Jan 15th, 2021.** **Time : 9.30 - 12.30.**

Maximum Points : 100. **Attempt as many questions as you can.**

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1. (10 points) Let G_n converge in probability in the local weak sense to (G, o). Let $(o_n^{(1)}, o_n^{(2)})$ be two independent uniformly chosen vertices in $[n]$. Show that $((G_n, o_n^{(1)}), (G_n, o_n^{(2)})) \xrightarrow{d} ((G, o), (G', o'))$ where (G', o') is an independent copy of (G, o).

2. (10 points) We define an end of a rooted tree (T, o) as a self-avoiding, semi-infinite path on T starting at o . Let (G, o) be a.s. an infinite connected and unimodular random graph. Show that if $\mathbb{E}[deg_G(o)] = 2$ then show that (G, o) is a.s. a tree and it has one or two ends a.s..

3. (10 points) Let $\mathbb{W}$, a Polish space be the mark space. Define the 'local weak topologies' on $\mathcal{G}_*(\mathbb{W}), \mathcal{G}_{**}(\mathbb{W})$ and show that both are complete seperable metric spaces.

4. (10 points) Construct the simplest (in your opinion) possible example where the local weak limit of a sequence of deterministic graphs is random.

5. (10 points) Let $K_{n,n}$ be the complete bi-partite graph on $2n$ vertices defined as follows : $V_n = [2n]$ is the vertex set and $E_n = \{(i, j) : 1 \leq i \leq n, n+1 \leq j \leq 2n\}$ is the edge-set. We define $G_n := G(n, p_n)$ the bi-partite random graph by choosing each edge in E_n independently of each other with probability $p_n = \min\{\lambda/n, 1\} \in [0, 1]$. What is the limit of G_n under local weak convergence ? Give a sketch of proof to justify the limit i.e., outline the main steps involved in the proof and provide some details on how you will prove the steps. You are not required to give a complete proof of the steps.

6. (10 points) If G is a graph with n vertices, then show that $s(F, G)$, subgraph density of F in G, changes by at most $\frac{k(k-1)}{n(n-1)}$ if one edge is changed.

7. Let $\mathcal{U}$ be the set of all unlabelled graphs. Consider the map $\tau : \mathcal{U} \to [0, 1]^{\mathcal{U}}$ defined by
$$\tau(G) := (s(F, G))_{F \in \mathcal{U}}.$$
Show that
   (a) (10 points) Show that $[0, 1]^{\mathcal{U}}$ is a compact metric space.
   (b) (5 points) Show that τ is not injective.
   (c) (10 points) Show that $\tau^+ : \mathcal{U} \to [0, 1]^{\mathcal{U}} \times [0, 1]$ defined by
   $$\tau^+(G) = (\tau(G), v(G)^{-1})$$
   is injective.

8. (15 points) Let H be an exchangeable random infinite graph in $\mathcal{L}_\infty$. Show that the following are equivalent
   (a) H is exchangeable
   (b) $H|_{[k]}$ has a distribution invariant under all permutations of $[k]$ for all $k \geq 1$.